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Note on the Preceding Paper.

By William E. Story.

In the foregoing valuable paper on the "*Geographical Problem of the Four Colours*," Mr. Kempe has substantially proved the fundamental theorem, which has been so long a desideratum, by a very ingenious method; but it seems desirable, to make the proof absolutely rigorous, that certain cases which are liable to occur, and whose occurrence will render a change in the formulae, as well as some modification of the method of proof, necessary, should be considered separately, and I have endeavoured to do this in the following note.

1. In the notation used on page 197, if, at any stage of the development, the patch next to be stripped off has no point of concourse or line on it, when it is stripped off an island will be disclosed, and

$$P' = P\,, \quad D' = D + 1\,, \quad B' = B + 1\,.$$

If the patch has no point of concourse but only a single line, when it is stripped off either a peninsula or a district with two boundaries (as in Fig. 15) will be disclosed; in the first case we have

$$P' = P + 1\,, \quad D' = D + 1\,, \quad B' = B + 2\,,$$

and in the second case

$$P' = P + 2\,, \quad D' = D + 1\,, \quad B' = B + 3\,.$$

These formulae hold only if the boundaries joined by the line on the patch counted as *two* (and not *one*, as in Figs. 16 and 1) before the patch was put on. If the patch have a point of concourse in which σ boundaries meet, and if the district disclosed when the patch is stripped off have β boundaries, we have

$$P' = P + \beta - 1\,, \quad D' = D + 1\,, \quad B' = B + \beta\,.$$

These equations are identical with those at the top of page 198, *i. e.* $\beta = \sigma$, only when three and no more boundaries meet in each point of concourse about the district patched out, for σ is evidently the number of boundaries meeting the boundaries of this district, each of the former counting for one or two according as it meets the latter in one or two *different* points.

In either of the above cases

$$P' + D' - B' - 1 = P + D - B - 1. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

But if the patch has no point of concourse but only a single line *forming part of the boundary of an island-district* on the patched map, so that when the patch is stripped off one of the forms of Figs. 16 and 1 is disclosed, we have for the former

$$P' = P + 2, \quad D' = D + 1, \quad B' = B + 2,$$

and for the latter (if the point in which the *two* boundaries meet be called a point of concourse)

$$P' = P + 1, \quad D' = D + 1, \quad B' = B + 1;$$

and hence for either of these cases

$$P' + D' - B' - 1 = P + D - B, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

instead of (a).

We will define a *contour* as an aggregate of boundaries such that, while any two of them are mutually connected, either directly or by means of other boundaries of the same contour, they are not connected with any other boundaries in the map. Such a contour will be *simple* or *complex* according as it consists of *one* or *more than one* boundary. Each contour may be considered as forming a map by itself, which may be coloured accordingly. In the process of patching out by Mr. Kempe's method the map formed by any *complex-contour*, we must arrive sooner or later at one of the forms of Figs. 16 and 1, next at an island, and then this disappears. In the reverse process we have at the first stage

$$P = 0, \quad D = 1, \quad B = 0,$$

i. e.

$$P + D - B - 1 = 0;$$

at the second stage, by (a),

$$P + D - B - 1 = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

at the third stage, by (b),

$$P + D - B - 1 = 1;$$

and at every subsequent stage, by (a),

$$P + D - B - 1 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Of course, in the case of a map formed by a *simple-contour* only the first and second stages exist, and for such a map equation (1) holds. If then $\varkappa$ of the contours formed by the boundaries of any map are complex, for that map

$$P + D - B - 1 = \varkappa. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

That is *in every map drawn on a simply connected surface the number of points of concourse and number of districts are together one greater than the number of* *boundaries and number of complex-contours together.* This takes the place of the theorem at the middle of page 198.

It is possible, as we have seen, that two boundaries of themselves form a point of concourse, as in Fig 1. Hence to the series $p_3, p_4, \ldots$ should be prefixed p_2, the number of points of concourse at the given stage of the development at which two boundaries meet.* Then

$$P = p_2 + p_3 + p_4 + \ldots$$

and

$$2B = 2\beta_0 + \beta_1 + 2p_2 + 3p_3 + 4p_4 + \ldots$$

Thus, since (3) may be written

$$(6D - 2B) + (6P - 4B) - 6(\varkappa + 1) = 0,$$

we have

$$5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - \ldots = 0,$$

as at the bottom of page 198.

2. At the top of page 199 the statement is made: "*if we develop a map so patched out, since each patch, when taken off, discloses a district with less than six boundaries, not more than five boundaries meet at the point of concourse on the patch.*" Now, as we have noticed above, the number of boundaries meeting in the point of concourse on a patch is equal to the number of boundaries of the district covered by the patch only when the number of boundaries meeting in each point of concourse about the district does not exceed three, and the statement quoted is therefore true only in this case. This difficulty may be obviated as follows:

Any point of concourse in which more than three boundaries meet may be removed, or replaced by a number of points of concourse in which only three boundaries meet, by sticking on it a small circular patch with a line drawn round it, of which those portions have been erased, which form the boundaries between the patch and any *one* of the districts bordering on it. Such a patch we will designate an *auxiliary-patch.* The number of districts in any map is not altered by an auxiliary-patch, since this forms only an extension of one of its districts over a point of concourse.

Having first *modified* the map by sticking an auxiliary-patch on each point of concourse in which more than three boundaries meet, we proceed to patch out the modified map as above, always putting a patch on a district with less than six boundaries, and sticking an auxiliary-patch on every point of concourse in which four or five boundaries meet, as it is produced (in this process a point of concourse in which more than five boundaries meet is never produced, nor are two auxiliary patches ever put on in immediate succession after the first district has been patched out). We thus finally arrive at a map containing one district and no boundary, which we colour with either one of four colours. Then, developing the map by stripping off the patches (including auxiliary patches) in the order inverse to that in which they were put on, we colour each district as it is disclosed. We will suppose that at a stage of the development, in which a certain district was disclosed, the map has been coloured with the four colours. The patch next to be stripped off will be either an ordinary patch with no line, or with one line and no point of concourse, or with a point of concourse in which not more than three boundaries meet; or it will be an auxiliary-patch. If it is an ordinary patch, when it is stripped off a district will be disclosed, on which border not more than three other districts, and at least one of the four colours will thus be at our disposal for the new district. If it is an auxiliary-patch, when it is stripped off a point of concourse will be disclosed, in which four or five boundaries, and not more than five districts, meet. The colours of these districts are to be extended over their uncovered portions, and the number of colours at the point of concourse reduced to not more than three by the method of page 195. The next patch (that on which is the point of concourse lately covered by the auxiliary-patch) will thus disclose, when stripped off, a district with four or five boundaries, surrounded by not more than five districts of not more than three different colours, and for which therefore at least one of the four colours will be at our disposal. Thus the map will be coloured with the four colours.

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* A point of concourse in which two boundaries meet counts for *one* in p_2 and also for *two* in β_1.

It is evidently unnecessary to extend the colours of the adjacent districts up to the point of concourse disclosed by an auxiliary-patch until we have disclosed the whole map as first *modified*, for a point of concourse appearing at an earlier stage will lie on a patch which has to be stripped off at the next stage.

BALTIMORE, *August 22d, 1879.*